



Regular embeddings of Cartesian product graphs

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ABSTRACT

It is proved in this paper that if a Cartesian power X^n of a prime graph X (with respect to the Cartesian multiplication) has an orientable regular embedding, then X has an orientable regular embedding too; and if a graph with some extra conditions has orientable regular embeddings, then its Cartesian power also has. As an application of our main theorems, the regular embeddings of grid-like graphs are studied.

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1. Introduction

All graphs considered in this paper are connected, undirected, finite and without loops or multiple edges. For a graph X , we denote by $V(X)$, $E(X)$, $D(X)$ and $\text{Aut}(X)$, the vertex set, the edge set, the arc set, and the automorphism group of X , respectively. For a positive integer n , we use $[n]$ to denote the set $\{1, \dots, n\}$. For the group and graph theoretical terminologies, see [4,1,6], respectively.

A *map* is a 2-cell embedding of a connected graph into a closed surface. The embedded graph is called the *underlying graph* of the map. An *orientable map* is such a map on an orientable surface. An *automorphism* of an orientable map is an automorphism of the underlying graph, which can be extended to an orientation-preserving homeomorphism of the supporting surface. It is well known that the automorphism group of an orientable map acts semi-regularly on the arc set of the underlying graph. If it acts regularly, the map is called *regular* as well. In this paper, we shall focus on the orientable regular maps, which are simply called regular maps.

An orientable map \mathcal{M} with underlying graph X can be described combinatorially by a pair $\mathcal{M} = (X; R)$, where R is a permutation of the arc set D whose orbits coincide with the sets of arcs emanating from the same vertex. The permutation R is called the *rotation* of the map \mathcal{M} . In the cycle decomposition of the permutation R , the cycle permuting the arcs emanating from a vertex v is called the *local rotation* R_v at v . Now, the map isomorphism and the map automorphism can be rephrased as follows: given two maps $\mathcal{M}_1 = (X_1; R_1)$ and $\mathcal{M}_2 = (X_2; R_2)$, a map isomorphism $\phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a graph isomorphism $\phi : X_1 \rightarrow X_2$ such that $R_1\phi = \phi R_2$. In particular, if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, ϕ is called a *map automorphism* of \mathcal{M} . All the automorphisms of \mathcal{M} form a group, say $\text{Aut}(\mathcal{M})$, called the automorphism group of map \mathcal{M} , acting always semi-regularly on the arc set D . If it acts regularly, the map is called regular.

It was shown in [5] that a graph X has a regular embedding if and only if its automorphism group $\text{Aut}(X)$ contains an arc-regular subgroup G such that the stabilizer G_u of each vertex u is cyclic. If $G = \langle a, b \rangle$ is an arc-regular subgroup of $\text{Aut}(X)$ with $G_u = \langle a \rangle$ for a fixed vertex u and b is an arc reversing automorphism for an arc emanating from u , then one can

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construct a regular map $\mathcal{M}(G; a, b) = \mathcal{M}(X; R)$ as follows: for any arc $(x, y) \in D(X)$, the local rotation R_x at x is defined by

$$(x, y)^{R_x} = (x, y)^{\psi_x^{-1} a \psi_x} = (x, y^{\psi_x^{-1} a \psi_x}),$$

where ψ_x is an automorphism in G mapping the vertex u to the vertex x . One can easily check that $\mathcal{M}(G; a, b)$ is indeed a well defined regular map with the underlying graph X . In this case, G is the automorphism group of $\mathcal{M}(G; a, b)$. If the orders of ba and a are s and t respectively, then $\mathcal{M}(G; a, b)$ has type $\{s, t\}$ in the notation of Coxeter and Moser [3], meaning that the faces are all s -gons and the vertices all have valency t .

Classification of regular maps by their underlying graphs is one of the central problems in topological graph theory. Of course, whether a graph has regular embeddings or not is also an interesting problem. As a matter of fact, not all graphs can be regularly embedded. Regularity alone is not sufficient to guarantee the existence of regular embeddings. Prism graphs \mathcal{P}_n for $n > 4$ (or $n = 3$), which are Cartesian products of cycle graphs C_n with path graph L_2 , are examples of regular graphs that cannot be regularly embedded in orientable surfaces (simply because they are not arc-transitive). But arc-transitivity does not suffice either. The complete graphs K_n for n not a power of a prime are arc-transitive and yet fail to have regular embeddings (see [7]).

In this paper, we study the existence of regular embeddings of the Cartesian products of regular graphs in orientable surfaces and obtain the following results: (1) a prime graph has orientable regular embeddings if its Cartesian power has; (2) if a graph with some extra conditions has orientable regular embeddings, then its Cartesian power also has. Our paper is organized as follows: some general statements on Cartesian product are introduced in Section 2; the main results are proved in Section 3; as an application of our main theorems, the regular embeddings of the grid-like graphs are studied in Section 4; and finally, some questions for further research are proposed in Section 5.

2. Cartesian product

The Cartesian product of two graphs X_1 and X_2 , denoted by $X_1 \square X_2$, is a graph with vertex set $V(X_1 \square X_2) = V(X_1) \times V(X_2)$ and edge set consisting of all pairs $\{(u_1, u_2), (v_1, v_2)\}$ such that $\{u_1, v_1\} \in E(X_1)$ and $u_2 = v_2$ or $u_1 = v_1$ and $\{u_2, v_2\} \in E(X_2)$. It is easy to verify that the Cartesian product is commutative and associative. Hence, we do not need parentheses for products of more than two factors. This allows an alternative direct definition of the Cartesian product of several factors. The Cartesian product $X = X_1 \square X_2 \square \cdots \square X_m$ of the graphs X_1, X_2, \dots, X_m is defined on the m -tuples (v_1, v_2, \dots, v_m) , where $v_i \in V(X_i)$, $1 \leq i \leq m$. Two m -tuples (u_1, u_2, \dots, u_m) and (v_1, v_2, \dots, v_m) are adjacent if there exists an index i such that $\{u_i, v_i\} \in E(X_i)$ and $u_j = v_j$ for all $j \in [m] \setminus \{i\}$.

Let U be the trivial graph given by $V(U) = \{u\}$ and $E(U) = \emptyset$. A graph X is called *prime* with respect to the Cartesian multiplication if X is non-trivial (i.e. if X is not isomorphic to U), and if $X \cong Y \square Z$ implies $Y \cong U$ or $Z \cong U$.

It was shown in [9] that every connected graph has a unique representation as a Cartesian product of prime graphs, up to isomorphisms and the order of the factors. Two graphs X and Y are called *relatively prime* with respect to the Cartesian multiplication if and only if they share no common factors in their Cartesian prime factorization.

Proposition 2.1 (Sabidussi, [9]). *The automorphism group of the Cartesian product of distinct connected prime graphs is isomorphic to the automorphism group of the disjoint union of the factors.*

Proposition 2.2 (Sabidussi, [9]). *Let X_1, X_2, \dots, X_m be pairwise relatively prime with respect to the Cartesian multiplication and $X = X_1 \square X_2 \square \cdots \square X_m$. Define the action of $\text{Aut}(X_1) \times \text{Aut}(X_2) \times \cdots \times \text{Aut}(X_m)$ on $V(X)$ by*

$$(x_1, x_2, \dots, x_m)^{(\xi_1, \xi_2, \dots, \xi_m)} = (x_1^{\xi_1}, x_2^{\xi_2}, \dots, x_m^{\xi_m}).$$

Then $\text{Aut}(X) = \text{Aut}(X_1) \times \text{Aut}(X_2) \times \cdots \times \text{Aut}(X_m)$.

Lemma 2.3. *If X is an arc-transitive graph, then X is either a prime graph or a Cartesian product of some isomorphic prime graphs.*

Proof. Suppose to the contrary. Let $X = Y \square Z$, where both Y and Z are non-trivial, and Y and Z are relatively prime with respect to Cartesian multiplication. By Proposition 2.2, $\text{Aut}(X) \cong \text{Aut}(Y) \times \text{Aut}(Z)$. Suppose that $(y_1, y_2) \in D(Y)$ and $(z_1, z_2) \in D(Z)$. Let $x_1 = (y_1, z_1)$, $x_2 = (y_2, z_1)$, and $x_3 = (y_1, z_2)$. Then both (x_1, x_2) and (x_1, x_3) belong to $D(X)$. But for each $(\eta, \zeta) \in \text{Aut}(Y) \times \text{Aut}(Z)$, $x_1^{(\eta, \zeta)} = x_1$ implies $y_1^\eta = y_1$ and $z_1^\zeta = z_1$. Hence $x_2^{(\eta, \zeta)} = (y_2, z_1)^{(\eta, \zeta)} = (y_2^\eta, z_1) \neq x_3$. Therefore $\text{Aut}(X) \cong \text{Aut}(Y) \times \text{Aut}(Z)$ is intransitive on $D(X)$, a contradiction. \square

For two nonempty sets A and B , we write $\text{Fun}(A, B)$ to denote the set of all functions from A to B . In the case where B is a group, we can turn $\text{Fun}(A, B)$ into a group by defining a product “pointwise”:

$$(\varphi\psi)(\alpha) = \varphi(\alpha)\psi(\alpha)$$

for all $\varphi, \psi \in \text{Fun}(A, B)$ and $\alpha \in A$ where the product on the right is in B .

The Cartesian product of n copies of X , denoted by X^n , is a graph defined on $\text{Fun}([n], V(X))$. Two functions f and g in $\text{Fun}([n], V(X))$ are adjacent if and only if there is a unique $i \in [n]$ such that $\{f(i), g(i)\} \in E(X)$ and $f(k) = g(k)$ for all $k \in [n] \setminus \{i\}$.

It is well known that (see [4]) $\text{Aut}(X) \wr S_n = \text{Fun}([n], \text{Aut}(X)) \rtimes S_n$, and the product action of $\text{Aut}(X) \wr S_n$ on $\text{Fun}([n], V(X))$ is defined by

$$f^{(\varphi, \sigma)}(k) = f(k^{\sigma^{-1}})^{\varphi(k^{\sigma^{-1}})}$$

for all $f \in \text{Fun}([n], V(X))$ and $(\varphi, \sigma) \in \text{Aut}(X) \wr S_n$.

Lemma 2.4. *With the above notations, we have:*

- (1) $\text{Aut}(X) \wr S_n \leq \text{Aut}(X^n)$;
- (2) *If X is a prime graph, then $\text{Aut}(X^n) = \text{Aut}(X) \wr S_n$.*

Proof. (1) For any $\{f, g\} \in E(X^n)$, there is a unique $i \in [n]$ such that $\{f(i), g(i)\} \in E(X)$ and $f(k) = g(k)$ for all $k \in [n] \setminus \{i\}$. Take $(\varphi, \sigma) \in \text{Aut}(X) \wr S_n$ with $i^\sigma = j$. Then

$$f^{(\varphi, \sigma)}(k) = f(k^{\sigma^{-1}})^{\varphi(k^{\sigma^{-1}})} \quad \text{and} \quad g^{(\varphi, \sigma)}(k) = g(k^{\sigma^{-1}})^{\varphi(k^{\sigma^{-1}})}.$$

Hence $f^{(\varphi, \sigma)}(k) = g^{(\varphi, \sigma)}(k)$ for all $k \in [n] \setminus \{j\}$. Since $\varphi(i) \in \text{Aut}(X)$, we have

$$\{f^{(\varphi, \sigma)}(j), g^{(\varphi, \sigma)}(j)\} = \{f(i)^{\varphi(i)}, g(i)^{\varphi(i)}\} \in E(X),$$

which implies $\{f^{(\varphi, \sigma)}, g^{(\varphi, \sigma)}\} \in E(X^n)$. Therefore, $\text{Aut}(X) \wr S_n \leq \text{Aut}(X^n)$.

- (2) Let $Y_i \cong X$ be n distinct graphs, where $i \in [n]$. Then $X^n \cong Y_1 \square Y_2 \square \cdots \square Y_n$. Suppose that Y is the disjoint union of Y_1, Y_2, \dots, Y_n . By Proposition 2.1, $\text{Aut}(X^n) \cong \text{Aut}(Y)$. Clearly, $\text{Aut}(Y) \cong \text{Aut}(X) \wr S_n$. Hence $\text{Aut}(X^n) = \text{Aut}(X) \wr S_n$. \square

3. Embeddings of Cartesian product graphs

A Cartesian product of some isomorphic graphs is called a *Cartesian power graph*. By Lemma 2.3, if a graph is neither a prime graph nor a Cartesian power of a prime graph, then it has no regular embeddings (simply because it is not arc-transitive). Therefore we only study the embeddings of Cartesian power graphs in this section.

Let X be a regular graph of order m and valency s . For brevity, we identify the vertex set of X with $[m]$. Then X^n has order m^n and valency ns ; and the vertex set of X^n is $\text{Fun}([n], [m])$. Let

$$U = \{f_1, f_2, \dots, f_m\},$$

where

$$f_i(k) = \begin{cases} i, & k = 1 \\ 1, & 2 \leq k \leq n \end{cases}, \quad i = 1, 2, \dots, m.$$

Suppose that Y is the induced subgraph of U in X^n . Then $\{f_i, f_j\} \in E(Y)$ if and only if $\{i, j\} \in E(X)$. Hence, $Y \cong X$.

Theorem 3.1. *If \mathcal{M} is a regular embedding of X^n and $\text{Aut}(\mathcal{M}) \leq \text{Aut}(X) \wr S_n$, then there is a regular embedding of X .*

Proof. Let $G = \text{Aut}(\mathcal{M})$. Since $G \leq \text{Aut}(X) \wr S_n$, an element in G can be written as (φ, σ) where $\varphi \in \text{Fun}([n], \text{Aut}(X))$ and $\sigma \in S_n$. Let

$$H = \{(\varphi, \sigma) \in G \mid 1^\sigma = 1\}.$$

Clearly, H forms a subgroup of G . For each $(\varphi, \sigma) \in H$ and each $f_i \in U$,

$$f_i^{(\varphi, \sigma)}(k) = \begin{cases} i^{\varphi(1)}, & k = 1 \\ 1^{\varphi(k^{\sigma^{-1}})}, & 2 \leq k \leq n. \end{cases}$$

Therefore, $f_i^{(\varphi, \sigma)} \in U$ if and only if $1^{\varphi(k)} = 1$ for all $k \in [n] \setminus \{1\}$. Let $K = H_{\{U\}}$, the setwise stabilizer of H . Then

$$K = \{(\varphi, \sigma) \in G \mid 1^{\varphi(k)} = 1, k = 2, \dots, n, 1^\sigma = 1\}.$$

Since $X \cong Y$, we need only to prove that K is an arc-regular subgroup of $\text{Aut}(Y)$ with cyclic vertex stabilizers. The proof is divided into three steps as follows:

- (1) Show the transitivity of K on U .

Choose an edge $\{f_i, f_j\} \in E(Y)$. Then $\{f_i, f_j\}$ is also an edge in $E(X^n)$. Since G is arc-transitive on X^n , there exists an element $(\beta, \lambda) \in G$, such that $f_i^{(\beta, \lambda)} = f_j$ and $f_j^{(\beta, \lambda)} = f_i$. Hence,

$$f_i(k^{\lambda^{-1}})^{\beta(k^{\lambda^{-1}})} = f_i^{(\beta, \lambda)}(k) = \begin{cases} j, & k = 1 \\ 1, & 2 \leq k \leq n \end{cases}$$

and

$$f_j(k^{\lambda^{-1}})^{\beta(k^{\lambda^{-1}})} = f_j^{(\beta, \lambda)}(k) = \begin{cases} i, & k = 1 \\ 1, & 2 \leq k \leq n. \end{cases}$$

It is easy to check that $1^\lambda = 1$ and $\beta(k) \in A_1$ for all $k \in [n] \setminus 1$. Hence $(\beta, \lambda) \in K$. Following the connectivity of Y , we get that K is transitive on U .

(2) Show the arc-transitivity of K on Y .

Let $G_{f_1} = \langle(\alpha, \rho)\rangle$. Then the order of (α, ρ) is ns and $1^{\alpha(k)} = 1$ for all $k \in [n]$. Suppose that t is the least integer such that $1^{\rho^t} = 1$. Let

$$\gamma = \alpha\alpha^{\rho^{-1}} \cdots \alpha^{\rho^{-(t-1)}}.$$

Then $t \leq n$, $1^{\gamma(k)} = 1$ and $(\alpha, \rho)^t = (\gamma, \rho^t)$. Hence, $\langle(\alpha, \rho)^t\rangle \leq K_{f_1}$. Since $G_{f_1} = \langle(\alpha, \rho)\rangle$ acts regularly on the neighbors of f_1 in X^n and $|\langle(\alpha, \rho)^t\rangle| = ns/t \geq s$, we get that $t = n$ and $\langle(\alpha, \rho)^t\rangle$ acts regularly on the neighbors of f_1 in Y . Therefore K_{f_1} is transitive on the neighbors of f_1 in Y . By the transitivity of K on U , we know that K is arc-transitive on Y .

(3) Show that K is regular on $D(Y)$ with cyclic vertex stabilizers.

For each arc $(f_i, f_j) \in D(Y)$, (f_i, f_j) is also an arc in $D(X^n)$. Noting that G is arc-regular on X^n , one gets that $(f_i, f_j)^{(\varphi, \sigma)} = (f_i, f_j)$ if and only if $(\varphi, \sigma) = 1$. Therefore K is arc-regular on Y with cyclic stabilizer $K_{f_1} = \langle(\alpha, \rho)^n\rangle$. \square

Remark 3.2. Suppose that $(1, i)$ is an arc of X . Then (f_1, f_i) is an arc of X^n . If $\mathcal{M}(G; (\alpha, \rho), (\beta, \lambda))$ is a regular embedding of X^n , where $G_{f_1} = \langle(\alpha, \rho)\rangle$ and (β, λ) is an involution transposing f_1 and f_i , then ρ is a n -cycle in S_n and λ is 1 or an involution in S_n fixing 1. Set $K = \langle(\gamma, 1), (\beta, \lambda)\rangle$ where $\gamma = \alpha\alpha^{\rho^{-1}} \cdots \alpha^{\rho^{-(n-1)}}$. Then $\mathcal{M}(K; (\gamma, 1), (\beta, \lambda))$ is a regular embedding of Y . Let $J = \langle\gamma(1), \beta(1)\rangle$. Then $\langle\gamma(1)\rangle$ cyclically permutes all neighbors of 1, and $\beta(1)$ is an involution transposing 1 and i . It is easy to check that the mapping

$$(\varphi, \sigma) \mapsto \varphi(1), \quad (\varphi, \sigma) \in K$$

gives an isomorphism from K to J . Hence J is an arc-regular subgroup of $\text{Aut}(X)$ with cyclic stabilizer $J_1 = \langle\gamma(1)\rangle$, from which follows that $\mathcal{M}(J; \gamma(1), \beta(1))$ is a regular embedding of X .

By Theorem 3.1 and Lemma 2.4, we have the following corollary.

Corollary 3.3. If a Cartesian power X^n of a prime graph X has a regular embedding, then X also has a regular embedding.

Conversely, we now consider the existence of regular embeddings of X^n provided X has a regular embedding. Without loss of generality, let $(1, 2)$ be an arc of X . Then (f_1, f_2) is an arc of X^n . From now on, we assume that $K = \langle a, b \rangle$ is an arc-regular subgroup of $\text{Aut}(X)$, where $K_1 = \langle a \rangle$ cyclically permutes all the neighbors of 1, and b is an involution transposing 1 and 2. Then $\mathcal{M}(K; a, b)$ is a regular embedding of X .

Let

$$\lambda_i(k) = \begin{cases} b, & k = i \\ 1, & k \in [n] \setminus \{i\}, \end{cases} \quad i = 1, \dots, n$$

Then $(\lambda_1, 1)$ is an involution transposing f_1 and f_2 .

For each $c \in A$, we use \bar{c} to denote the function in $\text{Fun}([n], A)$ where $\bar{c}(k) = c$ for all $k \in [n]$. Let

$$\mu(k) = \begin{cases} a, & k = 1 \\ 1, & 2 \leq k \leq n \end{cases}$$

and

$$\sigma = (12 \cdots n) \in S_n.$$

Then $(\mu, \sigma)^n = (\bar{a}, 1)$ and one can easily check that $\langle(\mu, \sigma)\rangle$ cyclically permutes all the neighbors of f_1 in X^n .

Theorem 3.4. Let $G = \langle(\mu, \sigma), (\lambda_1, 1)\rangle$. If $|\langle(\bar{a}, 1), (\lambda_1, 1)\rangle| \leq |K|$, then $\mathcal{M}(G; (\mu, \sigma), (\lambda_1, 1))$ is a regular embedding of X^n .

Proof. The theorem will be proved if we can show that G is an arc-regular subgroup of $\text{Aut}(X^n)$ with cyclic vertex stabilizers.

Clearly $\langle(\mu, \sigma)\rangle \leq G_{f_1}$. Since $(\lambda_1, 1)$ transposes the endpoints of the arc (f_1, f_2) and $\langle(\mu, \sigma)\rangle$ cyclically permutes all the neighbors of f_1 in X^n , by the connectivity of X^n , it follows that G is an arc-transitive subgroup of $\text{Aut}(X^n)$. In order to prove that G is an arc-regular subgroup of $\text{Aut}(X^n)$ with cyclic vertex stabilizers, we need only to show that $|G| = m^n ns$ (since the number of arcs of X^n is $m^n ns$).

Set

$$H = \langle(\bar{a}, 1), (\lambda_1, 1), \dots, (\lambda_n, 1)\rangle, \quad H_i = \langle(\bar{a}, 1), (\lambda_i, 1)\rangle.$$

Then $H = \langle H_1, \dots, H_n \rangle \trianglelefteq G$ and $G = H \langle (\mu, \sigma) \rangle$. Clearly, the mapping

$$(\phi, 1) \mapsto \phi(i), \quad (\phi, 1) \in H_i$$

gives an epimorphism from H_i to K for each $i \in [n]$. Since $|H_i| = |H_1| \leq |K|$, we have $H_i \cong K$. For $i, j \in [n]$ with $i \neq j$, take $(\varphi_i, 1) \in H_i$ and $(\varphi_j, 1) \in H_j$. Let

$$\psi_i = (\overline{\varphi_j(i)})^{-1} \varphi_i \overline{\varphi_j(i)}$$

and

$$\psi_j = \overline{\varphi_i(j)} \varphi_j (\overline{\varphi_i(j)})^{-1}.$$

Then $(\psi_i, 1) \in H_i$ and $(\psi_j, 1) \in H_j$. Since

$$(\varphi_i \varphi_j)(i) = \varphi_i(i) \varphi_j(i) = \varphi_j(i) (\varphi_j(i))^{-1} \varphi_i(i) \varphi_j(i) = \psi_j(i) \psi_i(i) = (\psi_j \psi_i)(i),$$

$$(\varphi_i \varphi_j)(j) = \varphi_i(j) \varphi_j(j) = \varphi_i(j) \varphi_j(j) (\varphi_i(j))^{-1} \varphi_i(j) = \psi_j(j) \psi_i(j) = (\psi_j \psi_i)(j)$$

and

$$(\varphi_i \varphi_j)(k) = \varphi_i(k) \varphi_j(k) = \varphi_j(k) \varphi_i(k) = \psi_j(k) \psi_i(k) = (\psi_j \psi_i)(k), \quad k \in [n] \setminus \{i, j\},$$

we have

$$(\varphi_i, 1)(\varphi_j, 1) = (\varphi_i \varphi_j, 1) = (\psi_j \psi_i, 1) = (\psi_j, 1)(\psi_i, 1).$$

Hence $H_i H_j = H_j H_i$, and then $H = H_1 H_2 \cdots H_n$. Since

$$H_i \bigcap H_j = \langle (\bar{a}, 1) \rangle,$$

we get

$$|H| = \left(\prod_{1 \leq i \leq n} |H_i| / |\langle (\bar{a}, 1) \rangle| \right) |\langle (\bar{a}, 1) \rangle| = m^n s.$$

Clearly

$$H \bigcap \langle (\mu, \sigma) \rangle \geq \langle (\bar{a}, 1) \rangle.$$

Therefore

$$|G| = |H \langle (\mu, \sigma) \rangle| = (|H| |\langle (\mu, \sigma) \rangle|) / |H \bigcap \langle (\mu, \sigma) \rangle| \leq m^n ns.$$

By the arc-transitivity of G , we have $|G| \geq m^n ns$. Hence $|G| = m^n ns$. \square

Theorem 3.5. Suppose that K has a normal subgroup N acting regularly on the vertex set of X . Let $G = \langle (\varphi, 1), (\mu, \sigma) | \varphi \in \text{Fun}([n], N) \rangle$. Then X^n has a regular embedding with G as its automorphism group.

Proof. Let $T = \{(\varphi, 1) | \varphi \in \text{Fun}([n], N)\}$. Since N is regular on $V(X)$ and $(\mu, \sigma) \in G_{f_1}$, we have T is regular on $V(X^n)$ and $T \bigcap \langle (\mu, \sigma) \rangle = 1$. Clearly

$$(\mu, \sigma)(\varphi, 1) = (\mu \varphi^{\sigma^{-1}}, \sigma) = (\mu \varphi^{\sigma^{-1}} \mu^{-1}, 1)(\mu, \sigma).$$

Hence

$$(\mu, \sigma)(\varphi, 1)(\mu, \sigma)^{-1} = (\mu \varphi^{\sigma^{-1}} \mu^{-1}, 1).$$

Since

$$\mu \varphi^{\sigma^{-1}} \mu^{-1} = \begin{cases} a \varphi(1^\sigma) a^{-1}, & k = 1 \\ \varphi(k^{\sigma^{-1}}), & 2 \leq k \leq n \end{cases}$$

and

$$a^{-1} N a = N,$$

we get $T \trianglelefteq G$, from which follows that $G = T \langle (\mu, \sigma) \rangle$. Hence G is arc-regular and the stabilizer $G_{f_1} = \langle (\mu, \sigma) \rangle$, a cyclic group. Correspondingly, X has a regular embedding with G as its automorphism group. \square

Recently, Jones [8] classified the regular embeddings of Hamming graphs. A *Hamming graph* is a Cartesian product of some isomorphic complete graphs. We use $H(d, n)$ to denote the Cartesian product of d copies of the complete graph K_n . By using our theorems obtained above, a new proof for one of Jones' results is given below.

Theorem 3.6 (Jones, [8]). $H(d, n)$ has regular embeddings if and only if n is a prime power.

Proof. It is well known (see [7]) that K_n has regular embeddings if and only if n is a prime power. Let \mathcal{M} be a regular embedding of K_n where $n = p^m$ is a prime power. As shown in [2], $\text{Aut}(\mathcal{M})$ has a normal subgroup acting regularly on the vertex set of K_n . Clearly, all complete graphs are prime graphs with respect to the Cartesian multiplication. Since $H(d, n) = (K_n)^d$, by Corollary 3.3 and Theorem 3.5, we have that $H(d, n)$ has regular embeddings if and only if n is a prime power. \square

4. Grid-like graphs

A *grid-like graph* is a Cartesian product of cycles. Clearly, a cycle is prime if and only if it is not a 4-cycle, while the 4-cycle is a Cartesian product of two copies of K_2 . Hence two cycles with different order are relatively prime with respect to the Cartesian multiplication.

Let $m > 2$. Then by C_m we denote the m -cycle. Write

$$V(C_m) = \{0, 1, \dots, m-1\},$$

where $\{i, j\} \in E(C_m)$ if and only if $|i - j| = 1$. Suppose that a and b are two permutations on $V(C_m)$ such that

$$i^a \equiv i + 1 \pmod{m} \quad \text{and} \quad i^b \equiv -i \pmod{m}.$$

Then the automorphism group of C_m is

$$\mathbb{D}_{2m} = \langle a, b \mid a^m = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

By Lemma 2.4, we have $\text{Aut}(C_m^n) = \begin{cases} \mathbb{Z}_2 \wr S_{2n}, & m = 4 \\ \mathbb{D}_{2m} \wr S_n, & m \neq 4. \end{cases}$

Clearly, $\mathcal{M}_m = \mathcal{M}(\mathbb{D}_{2m}; b, ba)$ is a regular embedding of C_m on the sphere and \mathbb{D}_{2m} has a normal subgroup $\langle a \rangle$ acting regularly on $V(C_m)$.

Now we give two examples of regular embeddings of C_m^n as follows:

Example 4.1. Take $f_i \in \text{Fun}([n], V(C_m))$ and $\mu, \lambda \in \text{Fun}([n], \mathbb{D}_{2m})$ where

$$f_i(k) = \begin{cases} i, & k = 1 \\ 0, & 2 \leq k \leq n \end{cases}, \quad i = 0, 1, \dots, m-1,$$

$$\mu(k) = \begin{cases} b, & k = 1 \\ 1, & 2 \leq k \leq n \end{cases}$$

and

$$\lambda(k) = \begin{cases} ba, & k = 1 \\ b, & 2 \leq k \leq n. \end{cases}$$

Let $\sigma = (12 \cdots n) \in S_n$. Then $f_0^{(\mu, \sigma)} = f_0$ and $\langle (\mu, \sigma) \rangle$ cyclically permutes all the neighbors of f_0 . Let

$$G = \langle (\varphi, 1), (\mu, \sigma) \mid \varphi \in \text{Fun}([n], \langle a \rangle) \rangle.$$

By Theorem 3.5, C_m^n has a regular embedding with G as its automorphism group. It follows that G is an arc-regular subgroup of $\text{Aut}(C_m^n)$ with cyclic stabilizer $G_{f_0} = \langle (\mu, \sigma) \rangle$. It is easy to check that

$$(\lambda, 1) \in G \quad \text{and} \quad (f_0, f_1)^{(\lambda, 1)} = (f_1, f_0).$$

Therefore $G = \langle (\mu, \sigma), (\lambda, 1) \rangle$. Correspondingly, $\mathcal{M} = \mathcal{M}(G; (\mu, \sigma), (\lambda, 1))$ is a regular embedding of C_m^n .

To compute the type of \mathcal{M} , let $\eta = \lambda\mu$. Then $(\lambda, 1)(\mu, \sigma) = (\eta, \sigma)$. Therefore

$$((\lambda, 1)(\mu, \sigma))^n = (\eta\eta^{\sigma^{-1}} \cdots \eta^{(\sigma^{n-1})^{-1}}, 1).$$

Since

$$\eta(k) = \lambda\mu(k) = \lambda(k)\mu(k) = \begin{cases} a^{-1}, & k = 1 \\ b, & 2 \leq k \leq n, \end{cases}$$

we have

$$\eta\eta^{\sigma^{-1}} \cdots \eta^{(\sigma^{n-1})^{-1}}(k) = \begin{cases} a^{-1}b^{n-1}, & k = 1 \\ b^{n-1}a^{(-1)^{k-1}}, & 2 \leq k \leq n. \end{cases}$$

Hence the order of $(\lambda, 1)(\mu, \sigma)$ is mn when n is odd and $2n$ when n is even. Since the order of (μ, σ) is $2n$, the type of \mathcal{M} is $\{mn, 2n\}$ if n is odd and $\{2n, 2n\}$ if n is even.

Example 4.2. Assume that m is even. Adopting the notations in Example 4.1, let

$$\tilde{\lambda}(k) = \begin{cases} ba, & k = 1 \\ 1, & 2 \leq k \leq n \end{cases} \quad \text{and} \quad \tilde{G} = \langle (\mu, \sigma), (\tilde{\lambda}, 1) \rangle.$$

Take $\tau \in \text{Fun}([n], \mathbb{D}_{2m})$ such that $\tau(k) = b$ for all $k \in [n]$. Let $H = \langle (\tau, 1), (\tilde{\lambda}, 1) \rangle$. Since $\tau\tilde{\lambda}(k) = \begin{cases} a, & k = 1 \\ b, & 2 \leq k \leq n \end{cases}$ and m is even, we have $((\tau, 1)(\tilde{\lambda}, 1))^m = (\tau\tilde{\lambda}, 1)^m = 1$. Therefore

$$(\tau, 1)^2 = (\tilde{\lambda}, 1)^2 = ((\tau, 1)(\tilde{\lambda}, 1))^m = 1.$$

It follows that $|H| \leq |\mathbb{D}_{2m}|$. By Theorem 3.4, $\tilde{\mathcal{M}} = \mathcal{M}(\tilde{G}; (\mu, \sigma), (\tilde{\lambda}, 1))$ is a regular embedding of C_m^n .

To compute the type of $\tilde{\mathcal{M}}$, let $\pi = \tilde{\lambda}\mu$. Then $(\tilde{\lambda}, 1)(\mu, \sigma) = (\pi, \sigma)$. Therefore

$$((\tilde{\lambda}, 1)(\mu, \sigma))^n = (\pi\pi^{\sigma^{-1}} \cdots \pi^{(\sigma^{n-1})^{-1}}, 1).$$

Since

$$\pi(k) = \tilde{\lambda}\mu(k) = \begin{cases} a^{-1}, & k = 1 \\ 1, & 2 \leq k \leq n, \end{cases}$$

we have $\pi\pi^{\sigma^{-1}} \cdots \pi^{(\sigma^{n-1})^{-1}} = a^{-1}$ for all $k \in [n]$. Hence the order of $(\tilde{\lambda}, 1)(\mu, \sigma)$ is mn . Since the order of (μ, σ) is $2n$, the type of $\tilde{\mathcal{M}}$ is $\{mn, 2n\}$.

Remark 4.3. If m is even, then C_m^n has regular embeddings \mathcal{M} and $\tilde{\mathcal{M}}$ constructed in Examples 4.1 and 4.2, respectively. Since \mathcal{M} and $\tilde{\mathcal{M}}$ have different types, they are different maps up to map isomorphism.

5. Questions

In this section we propose some questions for further research.

- (1) Is it possible to have a regular prime graph X which has regular embeddings and yet its Cartesian power X^n fails to have regular embeddings?
- (2) The main results of Section 3 poses a natural question: how can the general results about regular embeddings of Cartesian products help to obtain new classification results? In particular, under which conditions on X one could classify all regular embeddings of X^n provided that the classification of all regular embeddings of X is known? Even partial answers to this question would be very interesting.
- (3) Classify the regular embeddings of grid-like graphs. Are there any regular embeddings of C_m^n different from our constructions in Section 4 up to isomorphism?
- (4) Our attention only focuses on the orientable regular maps in this paper. What about the non-orientable case?

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